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Communication through a quantum link

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A chain of interacting spin behaves like a quantum mediator (quantum link), which allows two distant parties that control the ends of the chain to exchange quantum messages. We show that over repeated uses without resetting the study of a quantum link can be connected to correlated quantum channels with a finite dimensional environment (finite memory quantum channel). Then, using coding arguments for such kinds of channels and results on mixing channels we present a protocol that allows us to achieve perfect information transmission through a quantum link.

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I. INTRODUCTION

Recently, increasing attention has been devoted to interacting quantum systems in order to accomplish communication tasks. In fact, their evolution by means of quantum interference effects naturally leads to information transfer from one to another part located in a different place. A paradigmatic example is a chain of interacting spins (or more broadly speaking a spin network)—see Ref. [1], and references therein. Here two distant parties (say the sender Alice and the receiver Bob) try to exchange quantum messages by operating on separate ends of a chain of interacting qubits. Therefore the chain behaves like a mediator of quantum information, or like a *quantum link*.

The information Alice sends through the link can get stuck into the link itself, thus resulting in imperfect transmission. The faithfulness of information transfer has been widely investigated by considering the link to be reset on each use either by means of some external control operating directly on the whole chain, or by means of some clever but costly "downloading" procedure [1]. A multiuse quantum communication scenario without resetting is intriguing as well. Actually, a spin chain without resetting has been proposed as a physical model for quantum channel with memory [2]. A preliminary study of such complex communication lines has been carried out in Ref. [3] by computing the transmission rates (i.e., the number of transferred qubits per unit time) of some simple multiuse protocols, and in Ref. [4] by focusing on the two channel uses scenario of some specific spin chain models. Moving from such arguments, we study here the asymptotic (large number of uses) behavior of a quantum link without resetting.

In particular, we shall establish a connection between quantum link communication and a special class of correlated quantum channels, the finite memory channels, that allows us to devise a new communication strategy. Indeed using coding arguments for finite memory channels and some results from mixing channels we present a protocol that allows us to achieve perfect information transmission through the quantum link. It results in the first (efficient) communication protocol for the multiuse scenario of spin chains.

The layout of the paper is the following. In Sec. II we introduce the notion of perfect memory channels and we discuss coding arguments for them. In Sec. III we present a general communication scheme through a quantum link. Then, in Sec. IV we perform an information flow analysis by using the coding arguments previously developed and results on mixing channels. Finally, in Sec. V we present a protocol that allows us to achieve perfect information transmission through a spin chain. Sec. VI is for conclusions.

II. PERFECT MEMORY CHANNELS

In the multiuse quantum communication scenario the sender of information Alice transmits (classical or quantum) messages to her intended receiver Bob by encoding them in the internal states of an (possibly infinitely long) ordered *sequence* of information carriers $X := x_1, x_2, \ldots$. The latter are described as identical quantum systems characterized by the Hilbert spaces $\mathcal{H}_{x_1}, \mathcal{H}_{x_2}, \ldots$ of the same dimension $d := \dim[\mathcal{H}_x]$. Owing to the noise that affects the communication, the messages received by Bob are a corrupted version of the input signals. This process is formally described by assigning a *multiuse quantum channel*, i.e., a collection $\mathcal{L} := \{\Lambda^{(n)} : n \in \mathbb{N}\}$ of completely positive trace-preserving (CPTP) maps [5] $\Lambda^{(n)}$ connecting the input states of the carriers with their output counterparts. Specifically for any positive integer n, $\Lambda^{(n)}$ is the transformation that operates on the density matrices ρ_X of the Hilbert space $\mathcal{H}_X^{(n)} = \mathcal{H}_{x_1} \otimes \cdots \otimes \mathcal{H}_{x_n}$ associated to the first n carriers of X, i.e.,

$$\Lambda^{(n)}: \quad \rho_X \to \Lambda^{(n)}(\rho_X), \tag{1}$$

under the minimal consistency requirement that the output $\Lambda^{(n-1)}(\rho_X^{(n-1)})$ associated with the first $n-1$ carriers should be obtained from Eq. (1) by taking the partial trace with respect to the nth carrier. In this context $\mathcal{L}$ is said to represent a *memoryless quantum channel* if for all n the transformation (1) can be described as a *tensor product* of the CPTP map

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$\Lambda := \Lambda^{(1)}$ that acts on the states of the first carrier, i.e.,

$$\Lambda^{(n)}(\rho_X) = \Lambda^{\otimes n}(\rho_X), \tag{2}$$

with $\Lambda^{\otimes n} := \Lambda \otimes \cdots \otimes \Lambda$. If Eq. (2) does not apply one says instead that $\mathcal{L}$ represents a *correlated* channel. Furthermore, one says that it represents a *memory* channel if the sequence $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$, possesses a causal structure (i.e., if for all n, the output states of the first nth carriers $x_1, x_2, \ldots, x_n$ do not depend upon the input states of the subsequent carriers) [2,6,7].

It is well known [8,9] that any CPTP map admits unitary dilations that allow one to represent it in terms of a unitary coupling with an external environment. In particular, for the nth element of $\mathcal{L}$ we can write

$$\Lambda^{(n)}(\rho_X) = \mathrm{Tr}_Y[U_{XY}(\rho_X \otimes \omega_Y)U_{XY}^{\dagger}], \tag{3}$$

where U_{XY} is a unitary that couples the n carriers' state ρ_X to the state ω_Y of a multiuse environment Y described by the Hilbert space $\mathcal{H}_Y^{(n)}$. Upon purification one can always choose ω_Y to be a pure vector $|\omega\rangle_Y$: when this happens the dilation (3) is said to be of Stinespring form [9] and it is unique up to some irrelevant isometry acting on the environment Y. The unitary dilations (3) can also be put in a one-to-one correspondence with the *operator sum* (or *Kraus*) representations of $\Lambda^{(n)}$ [10],

$$\Lambda^{(n)}(\rho_X) = \sum_{j=0}^{d_Y^{(n)}-1} K_j \rho_X K_j^{\dagger}, \tag{4}$$

where $d_Y^{(n)} := \dim[\mathcal{H}_Y^{(n)}]$ and $K_j := {}_Y\langle \xi_j | U_{XY} | \omega \rangle_Y$ with $\{|\xi_j\rangle_Y ; j = 0, \ldots, d_Y^{(n)}-1\}$ being an orthonornomal basis of $\mathcal{H}_Y^{(n)}$.

From the uniqueness of Stinespring representation [9] it follows that (apart from the trivial case of noiseless, or unitary, transformations), the memoryless channels are characterized by possessing unitary dilations in which the environment has a dimension which is exponentially growing in n (i.e., $\log_2[\dim \mathcal{H}_Y^{(n)}] = n \log_2[\dim \mathcal{H}_Y^{(1)}]$) or, equivalently, by possessing a (minimal) operator sum representations whose Kraus sets contain a number of elements which are exponentially growing in n. This same property typically holds also for memory channels with the important exception of the so-called *perfect memory channels* [2,6,7]. They are characterized by the property of admitting unitary dilations (3) in which the dimension of the environment $\mathcal{H}_Y^{(n)}$ is constant in n (the extremal case being represented by the noiseless channels in which $\dim[\mathcal{H}_Y^{(n)}] = 0$ for all n). Such class of channels may be extended to include all sequences of CPTP maps that have a representation with a finite upper bound on the dimension of the environment state. More specifically:

Definition 1. A multiuse quantum communication channel $\mathcal{L}$ defined by the sequence of CPTP maps $\Lambda^{(n)}$ is termed as a perfect memory (PM) channel if there exists a sequence of unitary representations (3) of the $\Lambda^{(n)}$s such that

$$\lim_{n\to\infty} \frac{1}{n} \log_2[d_Y^{(n)}] = 0, \tag{5}$$

where $d_Y^{(n)}$ are the dimension of the environmental Hilbert spaces $\mathcal{H}_Y^{(n)}$ that enter in the dilation. Equivalently, $\mathcal{L}$ is said PM if it admits a sequence of operator sum representations characterized by a number of elements $d_Y^{(n)}$, which satisfy Eq. (5).

Physically speaking the property (5) means that the size of the environment E does not grow fast enough to capture all the information that is sent through the channel (the latter being measured by the size of the carriers $\mathcal{H}_X^{(n)}$, which is exponential in n). Intuitively we thus expect that PM channels should allow for efficient communication between Alice and Bob. This was formalized in Ref. [6] by showing that PM channels have indeed optimal (classical and quantum) transmission rates, i.e., allow the transfer of $\log_2 d$ qubit per channel use in the asymptotic limit of large n.

In particular, one can verify that the following theorems hold:

Theorem 1. Let $\mathcal{L}$ be a multiuse PM channel operating on information carriers of dimension d and let $\{d_Y^{(n)} : n \in \mathbb{N}\}$ be the sequence satisfying Eq. (5). Then for sufficiently large n there exists a zero-error classical code $\mathcal{C}_X^{(n)}$ of size

$$|\mathcal{C}_X^{(n)}| \geq \frac{d^n}{(d_Y^{(n)})^2}, \tag{6}$$

corresponding to a bit transmission rate $R_C^{(n)} := \frac{1}{n}\log_2|\mathcal{C}_X^{(n)}|$ that converges to the optimal value $\log_2 d$ for $n \to \infty$.

Proof. Given a positive integer n, let K_j be the $d_Y^{(n)}$ Kraus operators associated with an nth element of $\mathcal{L}$, i.e., the CPTP map $\Lambda^{(n)}$ that operates on the first n carriers. A zero-error classical code that corrects the noise introduced by $\Lambda^{(n)}$ is a collection $\mathcal{C}_X^{(n)}$ of (orthonormal) codewords $|c_k\rangle_X \in \mathcal{H}_X^{(n)}$, which must obey the conditions

$${}_X\langle c_k | K_i^{\dagger} K_j | c_{k'} \rangle_X = \delta_{kk'} M_{ij}(k), \tag{7}$$

for all codewords $|c_k\rangle_X$, $|c_{k'}\rangle_X$, and for all i,j, with $M_{ij}(1), M_{ij}(2), \ldots$, being $(d_Y^{(n)})^2 \times (d_Y^{(n)})^2$ Hermitian matrices [5,11]. These conditions imply that the support of the output states for all codewords are orthogonal, and hence may be distinguished with zero probability of error. Suppose one to have found ℓ orthogonal codewords $|c_1\rangle_X, \ldots, |c_\ell\rangle_X$ satisfying the condition (7) (this is always possible for at least $\ell = 2$). Then an additional codeword $|c_{\ell+1}^{(n)}\rangle_X$ can be chosen such that

$${}_X\langle c_{\ell+1} | K_i^{\dagger} K_j | c_k \rangle_X = 0, \tag{8}$$

for all i,j and for all $k = 1, \ldots, \ell$. Such a state exists provided that the total number of vectors $K_i^{\dagger} K_j |c_k\rangle_X$ is less than or equal to the dimension d^n of the input space $\mathcal{H}_X^{(n)}$, i.e., $\ell (d_Y^{(n)})^2 \leq d^n$ (notice that, due to the subexponential character of $d_Y^{(n)}$, for PM channels this inequality can always be satisfied for some positive ℓ, if n is sufficiently large). Hence, we may continue the procedure until the set of codewords cannot be extended. That is, we can get a code with at least $d^n/(d_Y^{(n)})^2$ orthogonal codewords that can be transmitted with zero error. This corresponds to a rate (i.e. ratio of the faithfully transferred classical bits over the number of channel uses) larger than $\log_2 d - \frac{2}{n}\log_2 d_Y^{(n)}$ which converges to the maximum value $\log_2 d$ attainable when using d-dimensional information carriers. ∎

To construct a quantum error correcting code, we can utilize the result of Ref. [11] in building a quantum code from the existing zero-error classical code.

Theorem 2. Let $\mathcal{L}$ be a multiuse PM channel operating on information carriers of dimension d and characterized by the sequence $\{d_Y^{(n)}:n\in\mathbb{N}\}$ satisfying Eq. (5). Then for sufficiently large n there exists a zero-error quantum error correcting code $\mathcal{Q}_X^{(n)}$ of size

$$|\mathcal{Q}_X^{(n)}| \geq \frac{d^n}{(d_Y^{(n)})^4+(d_Y^{(n)})^2}, \tag{9}$$

corresponding to a qubit transmission rate $R_{\mathcal{Q}}^{(n)} := \frac{1}{n}\log_2|\mathcal{Q}_X^{(n)}|$ that converges to the optimal value $\log_2 d$ for $n\to\infty$.

Proof. Under the same definitions of Theorem 1, the conditions for an error correcting *quantum* code are given by [5,11]

$$_X\langle q_k|K_i^\dagger K_j|q_{k'}\rangle_X = \delta_{kk'}M_{ij}, \tag{10}$$

where now the quantum codewords $|q_k\rangle_X$ are a basis for the coding subset $\mathcal{Q}_X^{(n)}\subseteq\mathcal{H}_X^{(n)}$ and with M_{ij} a matrix that *does not* depend on the index k of the quantum codewords $|q_k\rangle_X$. Consider then the classical code $\mathcal{C}_X^{(n)}$ that we have constructed in the derivation of Theorem 1. Let us divide it in ℓ nonoverlapping subsets $\mathcal{C}_X^{(n)}(1),\ldots,\mathcal{C}_X^{(n)}(\ell)$. For each $k=1,\ldots,\ell$ define also the vector $|q_k\rangle_X$ to be a (proper) superposition of the classical codewords that belong to $\mathcal{C}_X^{(n)}(k)$. One can easily verify that the set $\{|q_1\rangle_X,\ldots,|q_\ell\rangle_X\}$ still satisfies the classical code conditions (7) with matrices $M_{ij}(k)$, which are convex convolutions of the previous ones. The idea is thus to select the partitions $\mathcal{C}_X^{(n)}(1),\ldots,\mathcal{C}_X^{(n)}(\ell)$ and the associated vectors $\{|q_1\rangle_X,\ldots,|q_\ell\rangle_X\}$ in such a way that the new matrices $M(k)$ will all be identical: if this happens the vectors $\{|q_1\rangle_X,\ldots,|q_\ell\rangle_X\}$ will automatically satisfy the condition (10). In the end the problem can thus be mapped into a convex optimization problem: invoking a theorem by the Radon [12] it is then possible to show that it admits a solution provided that $\ell[(d_Y^{(n)})^4+(d_Y^{(n)})^2]<d^n$ (as before this inequality makes sense if $\mathcal{L}$ is PM at least for large enough n). In this way we can get a code with $d^n/[(d_Y^{(n)})^4+(d_Y^{(n)})^2]$ orthogonal codewords that can be transmitted with zero error. Hence the rate $\log_2 d-\frac{1}{n}\log_2[(d_Y^{(n)})^4+(d_Y^{(n)})^2]$ can be attained. If the final term scales such that it vanishes in the asymptotic limit, then the channel is asymptotically noiseless. This applies to all cases where $d_Y^{(n)}$ is subexponential in n. ■

A. Decoding transformation for PM channels

Even though this is a straightforward application of quantum error correction procedures [5] it is useful to give a close look at decoding strategy associated with Theorem 2. By construction the $(d_Y^{(n)})^2\times(d_Y^{(n)})^2$ matrix M_{ij} of Eq. (10) is positive semidefinite. Take then O_{ij} the unitary matrix that diagonalizes M_{ij} and define the $d_Y^{(n)}$ operators $F_j:=\Sigma_i O_{ij}K_i$. They provide an alternative sum operator decomposition of $\Lambda^{(n)}$ and satisfy the orthogonality condition

$$_X\langle q_k|F_i^\dagger F_j|q_{k'}\rangle_X = \delta_{kk'}\delta_{ij}\lambda_j, \tag{11}$$

for all the vectors $|q_k\rangle_X$ that form a basis of $\mathcal{Q}_X^{(n)}$, with $\lambda_j \geq 0$ the eigenvalues of M_{ij}. By polar decomposition we can then write

$$F_jP = \sqrt{\lambda_j}U_jP = \sqrt{\lambda_j}P_jU_j, \tag{12}$$

with U_j being a unitary transformation, P being the projector on $\mathcal{Q}_X^{(n)}$ such that $P_j:=U_jPU_j^\dagger$. From Eq. (11) it follows that the projectors P_j are orthogonal, i.e., $P_iP_j=\delta_{ij}P_j$. Consider thus a generic state ρ_X of $\mathcal{Q}_X^{(n)}$, i.e., $P\rho_XP=\rho_X$. Equation (12) allows us to express the output state as

$$\Lambda^{(n)}(\rho_X) = \sum_j \lambda_jP_j[U_j\rho_XU_j^\dagger]P_j^\dagger, \tag{13}$$

which is explicitly written in block form thanks to the orthogonality conditions of the P_js. We can hence recover ρ_X through the following steps: first perform a projective measurement on $\Lambda^{(n)}(\rho_X)$ that distinguishes among the orthogonal subspaces of $\mathcal{H}_X^{(n)}$ associated with the P_j. With probability λ_j we will get the outcome j. Apply then the unitary rotation $U_j^\dagger$ to the projected state: independently from the measurement outcome the final state will be transformed in the input message ρ_X.

Before concluding this section we would like to spend a few words on the global evolution that X and Y undergo during the encoding-decoding stages: this will play an important role in the subsequent sections. Consider then $|\psi\rangle_X \in \mathcal{Q}_X^{(n)}$. Using the unitary representation (3) associated with the Kraus set K_j and exploiting the above identities we get

$$\begin{aligned} U_{XY}(|\psi\rangle_X\otimes|\omega\rangle_Y) &= \sum_j (K_j|\psi\rangle_X)\otimes|\xi_j\rangle_Y \\ &= \sum_J \sqrt{\lambda_j}(P_jU_j|\psi\rangle_X)\otimes|\zeta_j\rangle_Y, \end{aligned} \tag{14}$$

where $|\zeta_j\rangle_Y:=\Sigma_iO_{ij}^*|\xi_i\rangle_Y$ is an orthonormal set of $\mathcal{H}_Y^{(n)}$. Of course by taking the partial trace with respect to Y yields the final state of Eq. (13). What is interesting for us, however, is to observe that *before* the decoding stage, X and Y are, in general, entangled. Furthermore, we notice that due to the orthogonality condition of the P_j and $|\zeta_j\rangle_Y$, the vector (14) is automatically written in the Schmidt form.

III. COMMUNICATION BY A QUANTUM MEDIATOR

As anticipated in the Introduction quantum networks communication [1] can be seen as a particular instance of the communication scenario sketched in Fig. 1. Here the *mediator M* is a composite quantum object of *finite* dimension d_M, which is composed by three subsystems M_A, M_C, and M_B, which interact through some given Hamiltonian H. M acts as an effective quantum channel that connects two distant parties, the sender of information Alice and the receiver Bob who are supplied with the quantum registers A and B, respectively. The register A is assumed to be composed by a sequence of ordered memories $a_1,a_2,\ldots$. In the communication scenario we consider, Alice "writes" on A the quantum messages she wants to communicate to Bob.

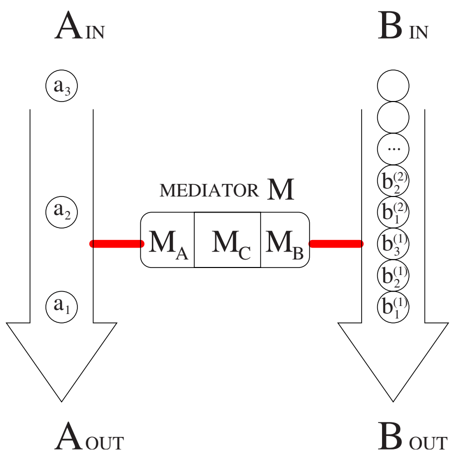

FIG. 1. (Color online) Communication through a quantum link. Alice sends to Bob the messages she has stored in the quantum memories $a_1, a_2, \ldots a_n$ by coupling them with the subsystem M_A of the mediator M; each memory element interacts only once with M_A following the sequential order indicated by the left arrow of the figure (i.e., first a_1, then a_2, and so on). Bob recovers Alice's messages by preparing each one of his quantum memories b in the same fixed fiduciary state $|\nu\rangle$ and by coupling them with the subsystem M_B of M (again the coupling will follow the sequential order indicated by the right arrow). The red lines of the figure represent the couplings between the registers and M. A_{in}, B_{in} represent the input ports of the device which are used by Alice and Bob to bring their qubits in contact with M. Similarly A_{out} and B_{out} represent the output ports from which the qubits emerge after their interaction with the mediator.

The memories $a_1, a_2, \ldots$ are then sequentially put in contact with the subsystem M_A of M following a fixed schedule in which first M_A couples with a_1, then with a_2, etc. Such interactions are assumed to be identical (*uniform coupling regime*), and faster and stronger than the free evolution of the mediator. Consequently, we will represent them in terms of a collection of two-body gates $S_{a_1}, S_{a_2}, \ldots, S_{a_n}$ that connect the corresponding memory elements and M_A. As indicated in Fig. 1 the whole process can be described as if the memories were moving in a line from the port A_{in} to the port A_{out} and interact with M_A only when passing close to it. Conversely, Bob's register B is composed by a sequence of memories each initialized in the same fiduciary state $\nu := |\nu\rangle\langle\nu|$. They are grouped in independent but not necessarily uniform subregisters $B^{(1)}, B^{(2)}, \ldots$. The kth subregister $B^{(k)}$ contains m_k memories indicated by the subscript $b_1^{(k)}, b_2^{(k)}, \ldots, b_{m_k}^{(k)}$, which are supposed to receive and store the info contained in Alice's kth input qubit a_k [13]. To do so, during the time interval that elapses between the Alice kth operation and the subsequent one, Bob will put the qubits of the subregister $B^{(k)}$ in contact with M by applying a series of fast quantum gates $S_{b_\ell^{(k)}}$ that couple the memories $b_\ell^{(k)}$ with the subsystem M_B (again one can represent this process as if Bob's memory elements were propagating from the port B_{in} of Fig. 1 to the port B_{out} interacting with M_B only when passing close to it). For each $k \in \mathbb{N}$ this yields the following unitary transformation acting on M and $B^{(k)}$:

$$V_k := S_{b_{m_k}^{(k)}} e^{-iH\tau} \cdots S_{b_2^{(k)}} e^{-iH\tau} S_{b_1^{(k)}} e^{-iH\tau}, \tag{15}$$

where $e^{-iH\tau}$ describes the free evolution of M between two consecutive quantum gates. In writing Eq. (15) we have assumed that, as in Alice's case, the couplings introduced by Bob are uniform and operate on a time scale much shorter than those of H. We assumed also that uniform time intervals τ elapse among any two consecutive $S_{b_\ell^{(k)}}$. The global unitary transformation of ABM during the transmission can then be expressed by composing the transformation V_k with Alice's gates S_{a_k}. Specifically, suppose that Alice uses only the first n elements of A for the communication and suppose that she prepares them in the state ρ_A. The initial state of the global system reads then $\rho_A \otimes \nu_B^{\otimes m} \otimes \omega_M$, with $m = \Sigma_{k=1}^n m_k$ being the total number of Bob's memories that play an active role in the transmission of the first n elements of A, $\nu_B^{\otimes m} := \nu_{b_1^{(1)}} \cdots \otimes \nu_{b_{m_n}^{(n)}}$ being their input state and with $\omega_M := |\omega\rangle_M\langle\omega|$ being the initial state of the mediator M that we will assume to be pure (the generalization to the mixed case being trivial). Bob's output states are thus given by

$$\Lambda_{A\to B}^{(n)}(\rho_A) = \mathrm{Tr}_{A,M}[W(\rho_A \otimes \nu_B^{\otimes m} \otimes \omega_M)W^\dagger], \tag{16}$$

with $\mathrm{Tr}_{A,M}$ being the partial trace with respect to A and M and with

$$W = V_n S_{a_n}, \ldots, V_2 S_{a_2} V_1 S_{a_1}. \tag{17}$$

IV. INFORMATION FLOW ANALYSIS

In this section we present a detailed analysis of the communication model of Fig. 1 that allows us to identify the two main mechanisms that superintend at the information flow through such a scheme. In particular, using the results on PM channels presented in Sec. II we will show that there are efficient, finite size encodings that allow one to prevent Alice messages from getting stuck into the mediator M. By itself this does not ensure perfect transfer from A to B. However, using ideas from the theory of mixing channels [14–17] we will show that one can force Alice messages to focus on Bob's memories.

Equation (16) defines a quantum channel $\Lambda_{A\to B}^{(n)}$, which connects the input port A_{in} of Fig. 1 to the output port B_{out}, i.e., which takes the input state of the n memories of the register A to the output state of the m memories of B. It is clearly a memory channel with M and A playing the role of the multiuse environment Y and with the memory effects arising from the possibility that part of the signals encoded in some earlier a_k will get stuck in M interfering with the subsequent ones. As mentioned in the Introduction, by identifying M with a network of interacting qubits, most of the spin communication protocols [1] can be represented in this model (an explicit example is presented in the next section). Having fixed the Hamiltonian H an interesting problem is then to determine wether or not there exist suitable choices of for the local transformations S_{a_k} and $S_{b_\ell^{(k)}}$, the timing τ, the encoding ρ_A, and possibly the fiduciary state ν of the B memories, which allows for a *reliable* and *efficient* information transmission from A_{in} to B_{out} (reliability referring to the possibility of achieving perfect transmission fidelity, efficiency referring instead to the effective number of memory elements—or coupling operations S_a—per transmitted qubit Alice needs to use). Even though one can easily find examples of H which admits simple answers for the above questions, in the general case this is not an easy problem to solve. Interestingly enough, however, the same questions admit an elegant solution if one considers the transmission efficiency of the channel $\Lambda^{(n)}_{A\to AB}$, which connects the input port A_{in} to the joint output ports $A_{out}+B_{out}$ of Fig. 1. In this case only M plays the role of the environment Y yielding the transformation

$$\Lambda^{(n)}_{A\to AB}(\rho_A) = \mathrm{Tr}_M[W(\rho_A \otimes \nu_B^{\otimes m} \otimes \omega_M)W^\dagger]. \tag{18}$$

Since M is a finite dimensional system (say of dimension d_M) the map $\Lambda^{(n)}_{A\to AB}$ is a perfect memory channel characterized by a Kraus set composed by a number of elements (i.e., d_M), which is constant in n. According to Sec. II, we know that there exists a zero-error quantum error correcting code $\mathcal{Q}_A^{(n)} \in \mathcal{H}_A^{(n)}$ of size $|\mathcal{Q}_A^{(n)}| \geqslant \frac{d^n}{d_M^2(d_M^2+1)}$ where d^n is the dimension of the register A (each memory having dimension d). Using such a code Alice can reliably transfer info from A_{in} to $A_{out}+B_{out}$ at a rate that is optimal for sufficiently large n (i.e., it allows one to transfer one qubit per memory element). In particular, consider the case of ω_M pure (the mixed case can be treated analogously upon purification) and indicate with P the projector onto $\mathcal{Q}_A^{(n)}$. According to the analysis of Sec. II A the global final state associated to a generic input state $|\psi\rangle_A$ of the coding subspace $\mathcal{Q}_A^{(n)}$ admits then the following Schmidt decomposition:

$$W(|\psi\rangle_A \otimes |\nu\rangle_B^{\otimes m} \otimes |\omega\rangle_M) = \sum_j \sqrt{\lambda_j}(P_jU_j|\psi\rangle_A \otimes |\nu\rangle_B^{\otimes m}) \otimes |\zeta_j\rangle_M, \tag{19}$$

where $\sqrt{\lambda_j}$ are Schmidt coefficients, U_j are unitary transformations acting on B and on the coding subspace of A, $P_j := U_jPU_j^\dagger$ are orthogonal projectors on AB, and $\{|\zeta_j\rangle_M; j=0,\dots,d_M-1\}$ is an orthonormal basis of M [5]. Therefore for all density matrices ρ_A of $\mathcal{Q}_A^{(n)}$ Eq. (18) can be expressed in the following block form:

$$\Lambda^{(n)}_{A\to AB}(\rho_A) = \sum_j \lambda_j P_jU_j(\rho_A \otimes \nu_B^{\otimes m})U_j^\dagger P_j, \tag{20}$$

while the recovery of the information can be obtained by performing a (possibly joint) projective measurement on AB, which distinguishes the orthogonal subspaces associated with the P_j.

Of course from the prospective of using the link M to transmit signals from A_{in} to B_{out} the above result is quite unsatisfactory: (i) it maps the messages into a possible joint subspace of A and B; and (ii) it requires one to perform joint operation on A and B to recover them. What is relevant for us, however, is the fact that using such encoding we can at least guarantee that Alice messages do not get stuck in M. Furthermore, the size of the quantum error correcting code $\mathcal{Q}_A$ is independent from the number m of Bob's qubits, as long as he prepares such qubits into a fixed reference state and couples them with M_B through a sequence of gates $S_{b_\ell^{(k)}}$ that are *known* to Alice. The real question is then determining whether or not one can force the output information to be localized only on B_{out}.

One indication that this may be possible for at least some spin network communication models considered so far [1] comes from a change of perspective. Consider in fact the channel that acts upon the mediator M between two consecutive interactions with Alice's qubits—say the kth and $(k+1)$th interactions. This is the map that given a state ω'_M of M takes it to

$$\mathrm{Tr}_{B^{(k)}}[V_k(\omega'_M \otimes \nu_{B^{(k)}}^{\otimes m_k})V_k^\dagger] = \underbrace{\mathcal{N}\circ\mathcal{N}\circ\cdots\circ\mathcal{N}}_{m_k}(\omega'_M),$$

where $\nu_{B^{(k)}}^{\otimes m_k} := \nu_{b_1^{(k)}} \otimes \cdots \otimes \nu_{b_{m_k}^{(k)}}$ is the input state of the kth subregister and $\circ$ represents superoperator composition, and where $\mathcal{N}$ is the CPTP map

$$\mathcal{N}(\omega'_M) = \mathrm{Tr}_b[(S_be^{-iH\tau})(\omega'_M \otimes \nu_b)(S_be^{-iH\tau})^\dagger]. \tag{21}$$

The above equations show that, under the repetitive interactions with the B memories, the evolution of the mediator M can be described as a sequence of iterated application of a channel. By general properties of CPTP maps it is known that in the asymptotic limit of $m_k\to\infty$ this will *typically* induce a relaxing behavior on M, which will bring such a system to a final point ω_M^* (the *fixed point* of the map) that is independent from the input state ω'_M [14–17]. This is known as the *mixing* or *relaxing* property of $\mathcal{N}$ and the associated convergency speed is known to the exponential fast in m_k (typically referring to the fact that the vast majority of CPTP maps possess the mixing property). We can hence use this result to say that for most choices of S_b and $|\nu\rangle_b$, by choosing $m_k \gg 1$ after any stage of Bob's coupling the system M can be brought *close* to a fix point ω_M^*. In turn this implies that the channel $\Lambda^{(n)}_{A\to A}$ that connects A_{in} to A_{out} will be *close* to the *memoryless* map $\tilde{\Lambda}^{\otimes n}_{A\to A}$ with $\tilde{\Lambda}_{A\to A}(\rho_A) := \mathrm{Tr}_M[S_A(\rho_A \otimes \omega_M^*)S_A^\dagger]$. For our purposes, however, the most appealing property of mixing channels is another one. Suppose in fact that the fix point ω_M^* of $\mathcal{N}$ is pure. In this case one can verify that, in the asymptotic limit $m_k\to\infty$ all information contained in M is transferred to B (the convergency speed being again exponentially fast in m_k). If the couplings S_A that connect A with M_A are then able to transfer Alice messages into M we can use the mixing property of $\mathcal{N}$ to *drive* such information into B.

V. SPIN CHAIN AS A QUANTUM LINK

In the following we introduce a spin chain implementation of the quantum link model of Fig. 1, which allows one to take full advantage of the analysis presented in the previous section. As a result will be able to show that one can use such spin chain to reliably transfer quantum information from A_{in} to B_{out}.

The setup we are interested in is similar to the one discussed in Ref. [18]: it uses as mediator M a collection of L identical, independent, N-long chains of $\frac{1}{2}$-spins. For $L=2$ it corresponds to the dual rail spin chain communication line of Ref. [19]: since the main properties of the model are already captured by the simpler case for the sake of simplicity in the following we will focus only on it (the case $L=1$ [20] has similar properties but it introduces some unnecessary complications in the discussion since it is lacking of a fundamental symmetry in the messages encoding). The spins of the two chains that form M are coupled through a (not necessarily first-neighbor) Heisenberg-like Hamiltonian H and are initially prepared into the *all-spin-down* configuration

$$|\omega\rangle_M \coloneqq \left| \begin{matrix} \downarrow\downarrow\cdots\downarrow \\ \downarrow\downarrow\cdots\downarrow \end{matrix} \right\rangle, \tag{22}$$

which one can assume to be the ground state of H (in writing the left-hand side of this expression we have adopted the simplified notation to represent the state of the first spin chain with the elements of the first line, and the state of the second spin chain with the elements of the second line). In this context the subsets M_A and M_B correspond, respectively, to the collection of the first spins and the last spins of the L chains. To simplify the analysis we also assume Alice's and Bob's memories to be qutrits characterized by the orthonormal states $|0\rangle$, $|1\rangle$, and $|E\rangle$: Alice encodes her input messages into the subspace $\{|0\rangle,|1\rangle\}^{\otimes n}$ of A generated by the components $|0\rangle$ and $|1\rangle$ (therefore each input memory of A will encode at most one qubit of info). Conversely we will identify $|E\rangle$ with the fiduciary state $|\nu\rangle$ of Bob's memories. For the couplings S_{a_k} and $S_{b_\ell^{(k)}}$ we chose mappings that act as the identity but for the following SWAP-like transformations,

$$|0\rangle \otimes \left| \begin{matrix} \downarrow \\ \downarrow \end{matrix} \right\rangle \leftrightarrow |E\rangle \otimes \left| \begin{matrix} \uparrow \\ \downarrow \end{matrix} \right\rangle,$$

$$|1\rangle \otimes \left| \begin{matrix} \downarrow \\ \downarrow \end{matrix} \right\rangle \leftrightarrow |E\rangle \otimes \left| \begin{matrix} \downarrow \\ \uparrow \end{matrix} \right\rangle, \tag{23}$$

where we used the same notation of Eq. (22) to represent the state of the spin chains. Notice that such couplings allow the transference of information from a_k to M_A only if both spins of M_A are pointing down: furthermore, when this happens a_k moves from the message subspace spanned by the vectors $|0\rangle$ and $|1\rangle$ to the state $|E\rangle$. If instead M_A has one or two spins up then the transmission is prevented and a_k and M_A keep their initial configurations. Similarly on Bobs side there is a net flow of information form M_B to $b_\ell^{(k)}$ only if the former has one spin up and one spin down: when this happens $b_\ell^{(k)}$ moves from the fiduciary state $|E\rangle$ to the message subspace and M_B will be promoted to the all spin down state. In all the other cases instead $b_\ell^{(k)}$ and M_B will keep their initial configurations. It is worth commenting that this is a main difference with respect to other swapping strategies introduced in the past (e.g., see Ref. [21]). In the present case in fact, there is no guarantee that the repetitive application of the gates $S_{b_\ell^{(k)}}$ will drive the chain toward the ground state (i.e., the associated map on M is not necessarily mixing). Such a property, however, still holds for at least the first excitation sector of the chains (this is the subspace of M in which both chains have exactly one spin up each): as will be clear in the following this is enough to set up a reliable transmission protocol.

A. Excitation distribution and Schmidt decomposition of the final state

We notice that both the couplings (23) and the spin chains Hamiltonian preserve a global observable Z, which counts the number of "excitations" present in the system, i.e.,

$$Z = Z_A + Z_B + Z_M, \tag{24}$$

where the operators Z_A and Z_B count, respectively, the number of memory elements of A and B that are in the subspace spanned by the message vectors $|0\rangle$ and $|1\rangle$ (i.e., the number of a_k and $b_\ell^{(k)}$, which are not in $|E\rangle$), and Z_M counts the number of spin up in the M. Furthermore, since at the beginning of the communication the state of the memories of A are in $\{|0\rangle,|1\rangle\}^{\otimes n}$, B is in $|E\rangle^{\otimes m}$, and M is in the all spin-down state (22), the value of Z during the whole protocol is set equal to n. Therefore the final state of the ABM system can be written as

$$W(|\psi\rangle_A \otimes |E\rangle_B^{\otimes m} \otimes |\omega\rangle_M) = \sqrt{\eta_0}|\Phi_\psi\rangle_{AB} \otimes |\omega\rangle_M + \sqrt{1-\eta_0}|\chi_\psi\rangle_{ABM}, \tag{25}$$

where η_0 is the probability to find M in the ground state $|\omega\rangle_M$. By construction the vector $|\Phi_\psi\rangle_{AB}$ is an eigenstate of Z_A+Z_B with eigenvalue n (i.e., it contains exactly n excitations) while $|\chi_\psi\rangle_{ABM}$ satisfies the following condition:

$${}_M\langle\omega|\chi_\psi\rangle_{ABM} = 0, \quad {}_{AB}\langle\Phi_{\psi'}|\chi_\psi\rangle_{ABM} = 0, \tag{26}$$

for all inputs $|\psi\rangle_A$ and $|\psi'\rangle_A$ (the first inequality follows from the fact that $|\chi_\psi\rangle_{ABM}$ has at least one excitation in M, the second from the fact that it contains strictly less than n excitations in AB). The quantity η_0 may, in general, depend on the input $|\psi\rangle_A$. However, it can be lower bounded by the joint probability that at each step of the protocol (i) Alice succeeds in inserting her qubit in M_A and (ii) Bob absorbs it in the associated subregister, i.e.,

$$\eta_0 \geqslant \prod_{k=1}^{n} p_k, \tag{27}$$

with p_k being the probability that at the kth step the info content of a_k moves to $B^{(k)}$. By induction one can easily convince oneself that the list of events associated with the probability $\Pi_n \coloneqq \prod_{k=1}^{n} p_k$ refers to a trajectory in which for the whole duration of the protocol M contains no more than one excitation per chain. By the previous discussion of the coupling (23) we know that for such states Bob's iterative procedure induces a relaxing behavior that drives the chains toward the ground state with a probability (i.e., p_k) that increases with m_k. Thus by choosing m_k sufficiently big we can make p_k and η_0 arbitrarily close to 1 (for instance by choosing $p_k \geqslant 1-\epsilon^k$ we get $\eta_0 \sim 1-\epsilon$). The probabilities p_k possess yet another important property: *they do not depend upon the information* $|\psi\rangle_A$ *encoded into the memories* a_k (i.e., the value of p_k is independent from the fact that a_k was in $|0\rangle_{a_k}$ or in $|1\rangle_{a_k}$). This is an extremely important property, which comes from the fact that we are using a dual rail encoding: it will allow us to simplify the whole analysis and will provide us a simple way to construct a working transmission protocol. Consider next the component $|\Phi_\psi\rangle_{AB}$ associated with η_0. By conservation of Z this state can be decomposed in two orthogonal terms as follows:

$$|\Phi_\psi\rangle_{AB} = \sqrt{\frac{\Pi_n}{\eta_0}}|E\rangle_A^{\otimes n} \otimes |\Phi_\psi\rangle_B + \sqrt{1-\frac{\Pi_n}{\eta_0}}|\Delta_\psi\rangle_{AB}. \quad (28)$$

The first component refers to the event associated with Π_n: Here all excitations of the input state have moved to B (whose state is an eigenvector of Z_B with eigenvalue n) leaving A into the state $|E\rangle_A^{\otimes n}$. It is worth stressing that for the same reason why Π_n is independent from the input state $|\psi\rangle_A$, then $|\Phi_\psi\rangle_B$ is a *faithful* encoding of such a vector. This can be expressed by saying that there exists an isometry from A to B, which connects $|\psi\rangle_A$ to $|\Phi_\psi\rangle_B$, or equivalently that there exists a unitary transformation U' on AB that is independent from $|\psi\rangle_A$ such that

$$U'(|\psi\rangle_A \otimes |E\rangle_B^{\otimes m}) = |E\rangle_A^{\otimes n} \otimes |\Phi_\psi\rangle_B. \quad (29)$$

A particular property of $|\Phi_\psi\rangle_B$ that is important to stress is the fact that *each one* of the registers $B^{(k)}$ that compose B contains exactly *one* excitation (specifically this is that same excitation that is initially contained in the kth memory cell of Alice). The second component of Eq. (28) contains elements orthogonal to $|E\rangle_A^{\otimes n}\otimes|\Phi_\psi\rangle_B$: These include terms that have at least one of the n excitations in A, plus terms that have no excitation in A but that have them in the "wrong" places of B with respect to the components $|E\rangle_A^{\otimes n}\otimes|\Phi_\psi\rangle_B$, i.e., they have not exactly one excitation in each one of the subregisters $B^{(k)}$. In particular, this implies that $|\Delta_\psi\rangle_{AB}$ is orthogonal to just the B component of $|E\rangle_A^{\otimes n}\otimes|\Phi_{\psi'}\rangle_B$ for all possible inputs $|\psi\rangle_A$ and $|\psi'\rangle_A$, i.e.,

$${}_B\langle\Phi_{\psi'}|\Delta_\psi\rangle_{AB} = 0. \quad (30)$$

Let us now compare Eq. (25) with Eq. (19), which only applies for input states $|\psi\rangle_A$ belonging to the subspace $\mathcal{Q}_A^{(n)} \subseteq \{|0\rangle,|1\rangle\}^{\otimes n}$, and which allows efficient communication from A_{in} to $A_{out}+B_{out}$ (once we have chosen the values of m_k such subspace always exists by the analysis of Sec. IV). From the orthogonality conditions of Eq. (26) and from the uniqueness of the Schmidt decomposition it follows that that there must exist a component of Eq. (19) (say the one corresponding to $j=0$), which coincides with the vector $\sqrt{\eta_0}|\Phi_\psi\rangle_{AB}\otimes|\omega\rangle_M$ (the only requirement being preventing η_0 from being degenerate—this can always be enforced by choosing $\Pi_n > 1/2$). This implies $\lambda_0 = \eta_0$, $|\zeta_0\rangle_M = |\omega\rangle_M$, and

$$|\Phi_\psi\rangle_{AB} = U_0(|\psi\rangle_A \otimes |E\rangle_B^{\otimes m}), \quad (31)$$

where we used the fact that $P_0U_0 = U_0P$ and that $|\psi\rangle_A \in \mathcal{Q}_A^{(n)}$. Putting this together with Eqs. (28) and (29) we get also that $|\Delta_\psi\rangle_{AB}$ must provide us with a unitary encoding of the input state, i.e., there must exist a unitary U'' on AB such that

$$|\Delta_\psi\rangle_{AB} = U''(|\psi\rangle_A \otimes |E\rangle_B^{\otimes m}) \quad (32)$$

(by linearity this in turn implies that the probability η_0 is constant for all $|\psi\rangle_A \in \mathcal{Q}_A^{(n)}$).

Another important thing we can learn from the comparison of Eq. (25) with Eq. (19) is obtained by taking the reduced density operator of AB. In particular, Eq. (25) yields the following block matrix:

$$\eta_0|\Phi_\psi\rangle_{AB}\langle\Phi_\psi| + (1-\eta_0)\mathrm{Tr}_M[|\chi_\psi\rangle_{ABM}\langle\chi_\psi|], \quad (33)$$

which for input states of $\mathcal{Q}_A^{(n)}$ must coincide with Eq. (20), which is also in block form. Since we have already identified $|\Phi_\psi\rangle_{AB}\langle\Phi_\psi|$ with the $j=0$ block of Eq. (20) this implies that the remaining term of Eq. (33) must fit on the $j \geq 1$ blocks of Eq. (20), i.e.,

$$(1-\eta_0)\mathrm{Tr}_M[|\chi_\psi\rangle_{ABM}\langle\chi_\psi|] = \sum_j \lambda_j P_j U_j(\rho_A \otimes |E\rangle_B\langle E|^{\otimes m})U_j^\dagger P_j. \quad (34)$$

Let us now take the projection of Eq. (33) into the subspace orthogonal to the one that supports the vectors $|E\rangle_A^{\otimes n} \otimes|\Phi_\psi\rangle_B$ (this is the subspace that contains all vectors that have strictly less than n excitation on B plus the vectors that have n excitations located in the *wrong places*). This is

$$\sigma_{AB} = \frac{\eta_0-\Pi_n}{1-\Pi_n}|\Delta_\psi\rangle_{AB}\langle\Delta_\psi| + (1-\eta_0)\mathrm{Tr}_M[|\chi_\psi\rangle_{ABM}\langle\chi_\psi|]. \quad (35)$$

The important observation here is to notice that notwithstanding the projection the information about the input state is still well preserved into AB. For the first component this is a consequence of Eq. (32) while for the second one this comes from Eq. (34).

B. Protocol

Now we have all elements to construct the protocol.

(1) The first step is to fix the numbers m_k of the memories that compose Bob's kth subregister $B^{(k)}$: the choice of these parameters is determined by how close to 1 we want Π_n of Eq. (27). This will determine the probability of success. For instance, let us assume that we select $\Pi_n > 1-\epsilon$ for some given $\epsilon > 0$ (in any case we will require $\epsilon < 1/2$ to make sure that η_0 is a nondegenerate Schmidt coefficient).

(2) Alice writes her messages into the quantum error correcting code $\mathcal{Q}_A^{(n)}$ of A, which is associated with the choice of m_k of the previous point.

(3) Alice and Bob start their sequences of repetitive operations in which the memories A and B are put in contact with M according to Eq. (23).

(4) At the end of the coupling stage the state of the system is as in Eq. (19). At this point Bob performs a two values projective measurement on B to check whether or not his memory B fits into the subspace that support the vectors $|\Phi_\psi\rangle_B$ of Eq. (28) (i.e., it projects B into a subspace that has exactly *one excitation in each of the subregisters* $B^{(k)}$). This measurement is analogous to the parity check of the dual rail protocol.

(5) From Eqs. (25) and (28) we know that if the result of the measurement is YES then the total system will be projected into the state

$$|E\rangle_A^{\otimes n} \otimes |\Phi_\psi\rangle_B \otimes |\omega\rangle_M, \tag{36}$$

which, thanks to Eq. (29), contains in B the input information sent by Alice (to recover it Bob needs only to perform a local operation on B). The YES outcome happens with probability Π_n.

(6) With probability $1-\Pi_n < \epsilon$ the local measure performed by Bob will produce the outcome NO projecting AB into the state of Eq. (35), which still retains the coherence of the input message.

Adopting the above procedure, with probability Π_n we can thus transfer more than $n-\log_2[d_M^2(d_M^2+1)]$ qubits of information from A_{in} to B_{out} by using n swap-in operation on Alice side. Without doing anything else, one can hence achieve an average transfer fidelity that is larger than Π_n and that can be made arbitrarily close to one by a proper choice of the numbers m_k of Bob's memories. How good is this result compared with other techniques? In effect the same average transfer fidelity is attainable without the PM encoding strategy by only adopting a pure dual rail downloading technique [19]. The relevance of the protocol however relies on the fact that, *even when Bob's measurement produces the NO outcome* (this happens with probability $1-\Pi_n$) *Alice messages are not completely lost*. In this case in fact, information still retains its coherence but it is "delocalized" among A_{out} and B_{out}. Alice and Bob can thus still try to get it back increasing the transmission fidelity to 1. In the following we present a couple of possible strategies that can be adopted to achieve this goal.

The easiest solution for compensating the NO event is to admit that Alice and Bob are provided with some shared e-bit. In this case when getting the NO outcome from his measurement Bob can ask Alice to teleport to him all her n memory elements. Since there are n qutrits this takes $n\log_2 3$ e-bit and $2n\log_2 3$ bits of classical communication from Alice to Bob. After teleportation Bob has direct access to the state Eq. (35), which still encodes perfectly the input message: to recover it he has only to perform a projective measurement that distinguishes the subspaces associated with the projectors P_j of Eq. (19) (notice that $|\Delta_\psi\rangle_{AB}$ is still within the P_0) and apply the proper unitary transformations. Considering that the probability of the NO outcome is $1-\Pi_n$ the average cost of the whole procedure will thus be $(1-\Pi_n)\times\log_2 3$ e-bit and $(1-\Pi_n)\times 2\log_2 3$ bits of classical communication from Alice to Bob per transmitted qubit [22].

An alternative solution relies in first depleting the link M from all its excitations resetting it to $|\omega_M\rangle$. This will require some sort of external (not necessarily coherent) control on M. It is worth noticing that since the messages are safely confined in the memories AB the resetting operation can be done without compromising them (in other communication scenarios [1] this will not be the case). Once more Alice can thus try to transfer to Bob the content of her memory qubits by adopting the same protocol we considered before, i.e., by recursively coupling them with M_A. It should be noticed however, that in order to do so she first has to *translate* the information contained in the memories A into the coding subspace $\mathcal{Q}_A^{(n)}$. Consider that after the first step of the protocol, each one of Alice memories a span a three-dimensional space generated by $|0\rangle$, $|1\rangle$, and $|E\rangle$ (compare this with the input state of the memories *before* the first transmission stage when Alice messages spanned only the subspace generated by $|0\rangle$ and $|1\rangle$). In total this corresponds to $n\log_2 3$ qubits that we need to transfer. Such a subspace can be fitted in the coding subspace $\mathcal{Q}_A^{(n_1)}$ of $n_1 = n\log_2 3 + \log_2[d_M^2(d_M^2+1)]$ new carriers (the extra term $\log_2[d_M^2(d_M^2+1)]$ being required to compensate with the fact that $\mathcal{Q}_A^{(n_1)}$ is a proper subspace of $\mathcal{H}_A^{(n_1)}$). Applying the transferring protocol to these new carriers will allow Bob to recover the messages with probability Π_{n_1}, and will increase the average fidelity from Π_n to $\Pi_n + (1-\Pi_n)\Pi_{n_1}$. The average number of channel uses employed in the process will instead be equal to $n+n_1(1-\Pi_n)$, corresponding to a rate $\frac{n}{n+n_1(1-\Pi_n)} \simeq \frac{1}{1+(1-\Pi_n)\log_2 3}$. Reiterating the whole procedure many times under the assumption of uniform success probabilities, $\Pi_n = \Pi_{n_1} = \Pi_{n_2} = \cdots = 1-\epsilon$, will give us unitary fidelity with asymptotic rate $\simeq 1-\epsilon\log_2 3$ (in deriving this expression we assumed $\epsilon\log_2 3 < 1$).

C. Two chains or one qubit and beyond

The protocol we discussed in the previous section uses $L=2$ parallel spin chains to transfer ~ 1 qubit of info per each channel use (i.e., per each coupling S_a), i.e., $\sim 1/2$ qubit per channel use per chain. Since each chain should be capable to transport one qubit of info per swap this is not very efficient, instead we would like to have ~ 1 qubit per channel use per chain. One way to attain such a rate is to replace the standard dual rail encoding [19] with the generalized dual rail encoding of Ref. [18] which, in the limit of $L \gg 1$ parallel chains, allows one to transfer ~ 1 one qubit per parallel chain. In this case the local operations (23) will be replaced by analogous coupling transformations, which couple the first (last) spins of the L chains with Alice (Bob) memory elements.

VI. CONCLUSION

In conclusion, we review the physics of PM channels by generalizing it to the case of the subexponential environment. We also established a connection between quantum link communication and PM, by presenting a general multiuse protocol that allows Alice and Bob to faithfully communicate through a spin chain quantum link without resetting it at each channel transmission. The protocol succeeds in faithfully transferring the messages with an arbitrarily high probability that can be tuned by means of Bob's operations. The protocol originates from the merging of two apparently distinct ideas: the codes for PM channels and the mixing property of quantum channels. This is a new approach, which in principle can be exploited in other contexts.

We believe that this paper paves the way for deeper studies on quantum links in communication scenarios. One possible direction to explore is the relation between the amount of resources needed and the invoked coupling Hamiltonian.

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